Robust Nonlocal Trace Spaces and Neumann Problems

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Joint work with Thorben Hensiek [6]

Setup

We consider nonlocal, symmetric operators of type

$$L_s u(x) := \lim_{\varepsilon\to 0^+} \int_{B_\varepsilon(x)^c} (u(x)-u(y))J_s(x,y)dy, \quad u:\mathbb{R}^d\to\mathbb{R} \text{ meas.}.$$

Here $s\in(0,1)$ and J_s is symmetric and comparable to the kernel of the fractional Laplacian, i.e. there exists a constant $\Lambda\geq 1$ independent of s such that

$$\Lambda^{-1}(1-s)\,|x-y|^{-d-2s} \leq J_s(x,y) \leq \Lambda(1-s)\,|x-y|^{-d-2s}. \quad (1)$$

- If $J_s(x,y)=\frac{2^{2s}s\Gamma(d/2+s)}{\pi^{d/2}\Gamma(1-s)}|x-y|^{-d-2s}$, then L_s is the fractional Laplacian $(-\Delta)^s$.
- The operators L_s localize to elliptic, second order operators as $s\to 1-$.

The notion of weak solutions to Dirichlet/Neumann problems for L_s is motivated by a Gauß-Green-type formula. Let $\Omega\subset\mathbb{R}^d$ be an open, bounded set and $u\in C_b^2(\mathbb{R}^d)$, $v\in C_b^1(\mathbb{R}^d)$, then

$$\int_\Omega L_s u(x)v(x)dx + \int_{\Omega^c} N_s u(x)v(x)dx = \mathcal{E}^s(u,v), \text{ where}$$

$$\mathcal{E}^s(u,v) := \frac{1}{2}\iint_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))(v(x)-v(y))J_s(x,y)dydx.$$

Here $N_s u(x) := \int_\Omega (u(x)-u(y))J_s(x,y)dy$ is the nonlocal normal derivative related to L_s and Ω. By (1), an appropriate function space for weak solutions is

$$V^s(\Omega\,|\,\mathbb{R}^d) := \{u:\mathbb{R}^d\to\mathbb{R} \text{ meas.}\,|\,[u,u]_{V^s(\Omega\,|\,\mathbb{R}^d)}<\infty\},$$
$$[u,v]_{V^s(\Omega\,|\,\mathbb{R}^d)} := \frac{1-s}{2}\iint_{(\Omega^c\times\Omega^c)^c}(u(x)-u(y))(v(x)-v(y))\,|x-y|^{-d-2s}dydx.$$

For bounded domains Ω this space equipped with the inner product $(u,v)_{V^s(\Omega\,|\,\mathbb{R}^d)} := (u,v)_{L^2(\Omega)}+[u,v]_{V^s(\Omega\,|\,\mathbb{R}^d)}$ is a separable Hilbert space.

- $V^s(\Omega\,|\,\mathbb{R}^d)\to H^1(\Omega)$ as $s\to 1-$.
- For which functions $g_s:\Omega^c\to\mathbb{R}$ does the Dirichlet/Neumann problem have a weak solution? What happens in the limit $s\to 1-$?

Question

Do there exist Hilbert spaces X^s of functions $g:\Omega^c\to\mathbb{R}$ for $s\in(0,1)$ such that

(i) there exist trace operators $\mathrm{Tr}: V^s(\Omega\,|\,\mathbb{R}^d)\to X^s$ which are continuous uniformly in the limit $s\to 1-$,

(ii) there exist extension operators $\mathrm{Ext}: X^s\to V^s(\Omega\,|\,\mathbb{R}^d)$ which are continuous uniformly as $s\to 1-$,

(iii) X_s converges to the classical trace space $H^{1/2}(\partial\Omega)$ as $s\to 1$?

Previous results on nonlocal trace and extension theorems

- Dyda, Kassmann '19 [3]
- Bogdan, Grzywny, Pietruska-Pałuba, Rutkowski '20 [1]
- Du, Tian, Wright, Yu '22 [2]
- Frerick, Vollmann, Vu '22 [5]

Literature

[1] Krzysztof Bogdan et al. "Extension and trace for nonlocal operators". In: *J. Math. Pures Appl. (9)* 137 (2020), pp. 33–69.

[2] Qiang Du et al. "Nonlocal trace spaces and extension results for nonlocal calculus". In: *J. Funct. Anal.* 282.12 (2022), Paper No. 109453, 63.

[3] Bartłomiej Dyda and Moritz Kassmann. "Function spaces and extension results for nonlocal Dirichlet problems". In: *J. Funct. Anal.* 277.11 (2019), pp. 108134, 22.

[4] Guy Foghem and Moritz Kassmann. "A general framework for nonlocal Neumann problems". In: *arXiv e-prints*, arXiv:2204.06793 (Apr. 2022), arXiv:2204.06793. arXiv: 2204.06793 [math.AP].

[5] Leonhard Frerick, Christian Vollmann, and Michael Vu. "The nonlocal Neumann problem". In: *arXiv e-prints*, arXiv:2208.04561 (Aug. 2022). arXiv: 2208.04561 [math.AP].

[6] Florian Grube and Thorben Hensiek. "Robust nonlocal trace spaces and Neumann problems". In: *arXiv e-prints*, arXiv:2209.04397 (Sept. 2022). arXiv: 2209.04397 [math.AP].

Trace / Extension results

Let $\Omega\subset\mathbb{R}^d$ be a bounded $C^{1,1}$-domain. We use the notation $d_x := \mathrm{dist}(x,\partial\Omega) := \inf\{|x-z|\,|z\in\partial\Omega\}$. For $f,g:\Omega^c\to\mathbb{R}$ we define

$$(f,g)_{L^2(\Omega^c,\tau_s)} := \int_{\Omega^c} f(x)g(x)\tau_s(x)dx \text{ with } \tau_s(x) := \frac{1-s}{d_x^s(1+d_x)^{d+s}}, \quad x\in\overline{\Omega}^c. \quad (2)$$

Additionally, we introduce the bilinear form

$$[f,g]_{\mathcal{T}^s(\Omega^c|\Omega^c)} := \int_{\Omega^c}\int_{\Omega^c}(f(x)-f(y))(g(x)-g(y))k_s(x,y)dxdy, \quad (3)$$

where

$$k_s(x,y) := \frac{(1-s)^2}{d_x^s(1+d_x)^s d_y^s(1+d_y)^s\big(|x-y|+d_x+d_y+d_x\,d_y\big)^d}, \quad x,y\in\overline{\Omega}^c. \quad (4)$$

We define the Hilbert space $\mathcal{T}^s(\Omega^c) := \{g:\Omega^c\to\mathbb{R} \text{ meas. } \mid \|g\|_{\mathcal{T}^s(\Omega^c)}<\infty\}$ endowed with the norm $\|g\|_{\mathcal{T}^s(\Omega^c)} := \big(\|g\|^2_{L^2(\Omega^c,\tau_s)}+[g,g]_{\mathcal{T}^s(\Omega^c|\Omega^c)}\big)^{1/2}$.

Theorem 1 *Let $s_\star\in(0,1)$. There exists a constant $C=C(d,\Omega,s_\star)>0$ such that for all $s\in(s_\star,1)$*

1. there exists a continuous trace operator $\mathrm{Tr}: V^s(\Omega\,|\,\mathbb{R}^d)\to\mathcal{T}^s(\Omega^c)$

$$\|\mathrm{Tr}\,u\|_{\mathcal{T}^s(\Omega^c)} \leq C\,\|u\|_{V^s(\Omega\,|\,\mathbb{R}^d)}, \quad u\in V^s(\Omega\,|\,\mathbb{R}^d),$$

2. there exists a continuous extension operator $\mathrm{Ext}: \mathcal{T}^s(\Omega^c)\to V^s(\Omega\,|\,\mathbb{R}^d)$

$$\|\mathrm{Ext}\,g\|_{V^s(\Omega\,|\,\mathbb{R}^d)} \leq C\,\|g\|_{\mathcal{T}^s(\Omega^c)}, \quad g\in\mathcal{T}^s(\Omega^c),$$

3. Tr *is the left inverse of* Ext, *i.e.* $\mathrm{Tr}\circ\mathrm{Ext}=\mathrm{id}$.

Theorem 2 *If $g\in H^1(\Omega^c)$, then*

$$\|g\|_{L^2(\Omega^c,\tau_s)} \to \|\gamma\,g\|_{L^2(\partial\Omega)},$$
$$[g,g]_{\mathcal{T}^s(\Omega^c|\Omega^c)} \to [\gamma\,g,\gamma\,g]_{H^{1/2}(\partial\Omega)}$$

as $s\to 1-$. Here $\gamma: H^1(\Omega^c)\to H^{1/2}(\partial\Omega)$ is the classical trace operator.

Neumann problems

We define the closed subspaces $V^s_\perp(\Omega\,|\,\mathbb{R}^d) = \{u\in V^s(\Omega\,|\,\mathbb{R}^d)\,|\,\int_\Omega u=0\}$ and $H^1_\perp(\Omega)=\{u\in H^1(\Omega)\,|\,\int_\Omega u=0\}$. The follows theorems extend results from [4].

Theorem 3 *Let $\{s_n\}\subset(0,1)$ be a sequence converging to 1 from below and J_{s_n} satisfy (1). Let $F_n\in V^{s_n}_\perp(\Omega\,|\,\mathbb{R}^d)'$, $G_n\in\mathcal{T}^{s_n}(\Omega^c)'$ such that $\sup_n\|F_n\|_{V^{s_n}_\perp\to\mathbb{R}}$, $\sup_n\|G_n\|_{\mathcal{T}^{s_n}\to\mathbb{R}}<\infty$ and $u_n\in V^{s_n}(\Omega\,|\,\mathbb{R}^d)$ be the weak solution to*

$$L_{s_n}u_n = F_n \text{ in } \Omega,$$
$$N_{s_n}u_n = G_n \text{ on } \Omega^c,$$
$$\text{i.e. } \mathcal{E}^{s_n}(u_n,v) = F_n(v)+G_n(\mathrm{Tr}\,v) \text{ for all } v\in V^{s_n}_\perp(\Omega\,|\,\mathbb{R}^d).$$

There exist $F\in H^1_\perp(\Omega)'$, $G\in H^{1/2}(\partial\Omega)'$ such that a subsequence $\{u_{n_k}\}$ converges in $L^2(\Omega)$ to the solution $u\in H^1_\perp(\Omega)$ to the Neumann problem

$$\mathrm{div}\,A(\cdot)\nabla u = F \text{ in } \Omega,$$
$$n\cdot(A(\cdot)\nabla u) = G \text{ on } \partial\Omega,$$
$$\text{i.e. } \mathcal{E}^A(u,v) := \int_\Omega \nabla u\cdot(A(\cdot)\nabla v) = F(v)+G(\gamma v) \text{ for all } v\in H^1_\perp(\Omega)$$

with $A=(a_{i,j})$ given by $a_{i,j}(x) := \lim_{s\to 1-}\frac{1}{2}\int_{B_1(0)} h_i h_j J_s(x,x+h)dh$. Furthermore,

$$\mathcal{E}^{s_{n_k}}(u_{n_k},v)\to\mathcal{E}^A(u,v|_\Omega) \text{ for all } v\in H^1(\mathbb{R}^d)\cap L^2_\perp(\Omega).$$

Theorem 4 *Let $A(\cdot):\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d$ be symmetric and satisfy $\lambda^{-1}|\xi|^2\leq(A(x)\xi)\cdot\xi\leq\lambda|\xi|^2$, $f\in L^2(\Omega)$, $g\in L^2(\partial\Omega)$ and $u\in H^1_\perp(\Omega)$ be the weak solution to*

$$\mathrm{div}\,A(\cdot)\nabla u = f \text{ in } \Omega,$$
$$n\cdot(A(\cdot)\nabla u) = g \text{ on } \partial\Omega.$$

Then there exist a sequence $\{s_n\}\subset(0,1)$ converging to 1, symmetric kernels J_{s_n} satisfying (1) and $g_n\in L^2(\Omega^c,\tau_{s_n}^{-1})$ such that the weak solutions $u_n\in V^{s_n}_\perp(\Omega\,|\,\mathbb{R}^d)$ to the nonlocal Neumann problem $L_{s_n}u_n=f$ in Ω and $N_{s_n}u_n=g_n$ on Ω^c converges to $u\in H^1_\perp(\Omega)$ in $L^2(\Omega)$. Furthermore,

$$\mathcal{E}^{s_n}(u_n,v)\to\mathcal{E}^A(u,v|_\Omega) \text{ for all } v\in H^1(\mathbb{R}^d)\cap L^2_\perp(\Omega).$$

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