Robust Nonlinear Nonlocal Trace Spaces

Joint work with Moritz Kassmann

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Setup

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. We want to solve the Dirichlet problem in the weak sense

$$\begin{cases} (-\Delta)_p^s u = 0 & \text{in } \Omega \\ u = g & \text{on } \Omega^c \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian for $s \in (0,1)$ and $1 < p < \infty$ given by

$$(-\triangle)_p^s u(x) := (1-s) \lim_{\varepsilon \to 0^+} \int\limits_{B_\varepsilon(x)^c} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{d+sp}} dy.$$

- For $p=2$ this coincides with the fractional Laplacian up to a constant.
- The operator $-(-\triangle)_p^s$ localizes to the p-Laplacian $\Delta^p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ as $s \to 1^-$.
- Euler-Lagrange equation associated with the functional $J_{s,p}(v) := \int\limits_{(\Omega^c \times \Omega^c)^c} |v(x)-v(y)|^p \,/\, |x-y|^{d+sp} \, \mathrm{d}(x,y)$.
- The appropriate function space for weak solutions is $V^{s,p}(\Omega \,|\, \mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} \text{ m.b.} \,|\, J_{s,p}(u) < \infty\}$.
- In [3] it has been proven that $V^{s,p}(\Omega \,|\, \mathbb{R}^d) \to W^{1,p}(\Omega)$ as $s \to 1^-$.

Goal: Find a sequence of Banach spaces $\mathcal{T}^{s,p}(\Omega^c)$ of functions on Ω^c such that the trace $V^{s,p}(\Omega \,|\, \mathbb{R}^d) \ni u \mapsto u|_{\Omega^c}$ is continuous and there exists a continuous right inverse $\text{Ext}_s : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d)$. Furthermore, can we pick the spaces such that $\mathcal{T}^{s,p}(\Omega^c) \to W^{1-1/p,p}(\partial\Omega)$ as $s \to 1^-$?

Related literature: [5, 2, 1, 4, 6]

Result

We define measures $\mu_s(\mathrm{d}x) := \mathbb{1}_{\Omega^c}(x)(1-s)d_x^{-s}(1+d_x)^{-d-s(p-1)}\mathrm{d}x$ on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$, $s \in (0,1)$, $p \geq 1$ where $d_x := \operatorname{dist}(x, \partial\Omega)$ for $x \in \mathbb{R}^d$. We introduce our trace spaces $\mathcal{T}^{s,p}(\Omega^c) := \{g : \Omega^c \to \mathbb{R} \text{ m.b.} \,|\, \|g\|_{\mathcal{T}^{s,p}(\Omega^c)} < \infty\}$ where

$$\|g\|_{\mathcal{T}^{s,p}(\Omega^c)}^p := \|g\|_{L^p(\Omega^c;\,\mu_s)}^p + [g]_{\mathcal{T}^{s,p}(\Omega^c)}^p, \qquad [g]_{\mathcal{T}^{s,p}(\Omega^c)}^p := \iint\limits_{\Omega^c\,\Omega^c} \frac{|g(x)-g(y)|^p}{((|x-y|+d_x+d_y) \wedge 1)^{d+s(p-2)}} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y).$$

Theorem 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$, $1 < p < \infty$. Then the trace map

$$\text{Tr}_s : V^{s,p}(\Omega \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,p}(\Omega^c), \quad u \mapsto u|_{\Omega^c}$$

is continuous and linear and there exists a continuous right inverse

$$\text{Ext}_s : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d), \quad g \mapsto \text{Ext}_s(g).$$

The continuity constants of both operators depend on Ω, a lower bound on s as well as a lower and upper bound on p.

Theorem 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. For any $u \in W^{1,p}(\mathbb{R}^d)$

$$\|\text{Tr}_s u\|_{L^p(\Omega^c;\,\mu_s)} \to \|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \text{ and } [\text{Tr}_s u]_{\mathcal{T}^{s,p}(\Omega^c)} \to [u|_{\partial\Omega}]_{W^{1-1/p,p}(\partial\Omega)} \text{ as } s \to 1^-.$$

Literature

[1] Krzysztof Bogdan et al. "Nonlinear nonlocal Douglas identity". In: *arXiv e-prints* (June 2020). arXiv: 2006.01932 [math.AP].

[2] Krzysztof Bogdan et al. "Extension and trace for nonlocal operators". In: *J. Math. Pures Appl. (9)* 137 (2020), pp. 33–69.

[3] Jean Bourgain, Haim Brezis, and Petru Mironescu. "Another look at Sobolev spaces". In: *Optimal control and partial differential equations*. IOS, Amsterdam, 2001, pp. 439–455.

[4] Qiang Du et al. "Nonlocal trace spaces and extension results for nonlocal calculus". In: *J. Funct. Anal.* 282.12 (2022), Paper No. 109453, 63.

[5] Bartłomiej Dyda and Moritz Kassmann. "Function spaces and extension results for nonlocal Dirichlet problems". In: *J. Funct. Anal.* 277.11 (2019), pp. 108134, 22.

[6] Florian Grube and Thorben Hensiek. "Robust nonlocal trace spaces and Neumann problems". In: *arXiv e-prints* (Sept. 2022). arXiv: 2209.04397.

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