The Dirichlet Problem for Lévy-stable operators with L^2-data

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Motivation

For suff. regular bd. domains $\Omega \subset \mathbb{R}^d$ and boundary data $g \in L^2(\partial\Omega)$ it is known that the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

admits a unique solution u in $H^{1/2}(\Omega)$.

Integro-differential operators like the fractional Laplacian require functions to be defined on $\mathbb{R}^d$. Boundary value problems turn into exterior value problems with prespribed data $g : \Omega^c \to \mathbb{R}$.

Goal: Existence, uniqueness, and regularity of solutions to exterior value problems with data in weighted $L^2(\Omega^c)$-spaces. *Desired feature*: recover the theory for (1).

Stable operators

We study $2s$-stable, nondegenerate integro-differential operators:

$$A_s u(x) := p.v. \int_{\mathbb{R}^d} (u(x) - u(x+h))\, \nu_s(dh)$$

with $s \in (0,1)$, a measure μ on the unit sphere $\mathbb{S}^{d-1}$, and the Lévy measure ν_s given in polar coordinates via

$$\nu_s(U) := (1-s) \int_{\mathbb{R}} \int_{\mathbb{S}^{d-1}} \mathbb{1}_U(r\theta) |r|^{-1-2s} \mu(d\theta) dr.$$

We assume that there exist $0 < \lambda \le \Lambda$ such that

$$\lambda \le \inf_{\omega \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\omega \cdot \theta|^{2s} \mu(d\theta) \le \mu(\mathbb{S}^{d-1}) \le \Lambda.$$

- Note that: $A_s u(x) \to a^{ij}\partial_{ij} u(x)$ in the limit $s \to 1-$.

Examples:

- The fractional Laplacian $(-\Delta)^s$ corresponds to $\mu = 1$, i.e. $\nu_s(h) = (1-s)|h|^{-d-2s}$.
- The choice $\mu = \delta_{e_1} + \delta_{e_2}$ leads to $(-\partial^2_{x_1})^s + (-\partial^2_{x_2})^s$.

Exterior data

The exterior data will be taken from $L^2(\Omega^c; \tau_s)$, where

$$\tau_s(x) := \frac{1-s}{\text{dist}(x,\Omega)^s} \mathbb{1}_{\Omega^1 \cap \text{supp}(\nu_s^\star)}(x) + \nu_s^\star(x),$$

$\Omega^1 := \{y \in \Omega^c \mid \text{dist}(y,\Omega) < 1\}$, and $\nu_s^\star$ is the tail-weight adapted to ν_s

$$\nu_s^\star(x) := (1-s) \int_{\mathbb{R}} \int_{\mathbb{S}^{d-1}} \mathbb{1}_\Omega(x + r\theta)(1+|r|)^{-1-2s} \mu(d\theta) dr.$$

In [GK23]: $L^2(\Omega^c;\tau_s) \to L^2(\partial\Omega)$ as $s \to 1-$.

Result

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,\alpha}$-domain, $\alpha \in (0,1)$, $s_\star \in (0,1)$. For any $s \in (s_\star, 1)$, $f \in L^2(\Omega; d_x^{2s})$, $g \in L^2(\Omega^c; \tau_s)$, there exists a unique distributional solution $u \in H^{s/2}(\Omega) \cap L^2(\mathbb{R}^d; \nu_s^\star)$ to*

$$\begin{aligned} A_s u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Omega^c. \end{aligned} \tag{2}$$

Furthermore, the estimate

$$\|u\|_{H^{s/2}(\Omega)} \le C\Big(\|f\|_{L^2(\Omega; d_x^{2s})} + \|g\|_{L^2(\Omega^c;\tau_s)} \Big)$$

holds with some constant $C = C(d, \Omega, \alpha, s_\star, \lambda, \Lambda)$.

Main difficulties

- The low regularity of g prohibits the use of variational methods, i.e. functions in $L^2(\Omega^c;\tau_s)$ do not extend to functions in the energy space associated with A_s.
- Estimates of the Poisson kernel associated with A_s on $C^{1,\alpha}$-domains are not known.
- The connection between the equation (2) and the $H^{s/2}$ seminorm is not obvious. In particular, we need a robust in s, weighted Sobolev inequality.

References

[GHS23] F. Grube, T. Hensiek, and W. Schefer. "The Dirichlet Problem for Lévy-stable operators with L^2-data". In: *arXiv e-prints* (July 2023). arXiv: 2307.15235.

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